Homotopy of Paths

Definition A loop (or a closed loop) in a space X is a continuous function $\alpha : I = [0,1] \to X$ such that $\alpha(0) = \alpha(1)$, and we shall say that the loop is based at the point $\alpha(0)$.

If α and β are two loops based at the same point of X, we define the product $\alpha * \beta$ to be the loop

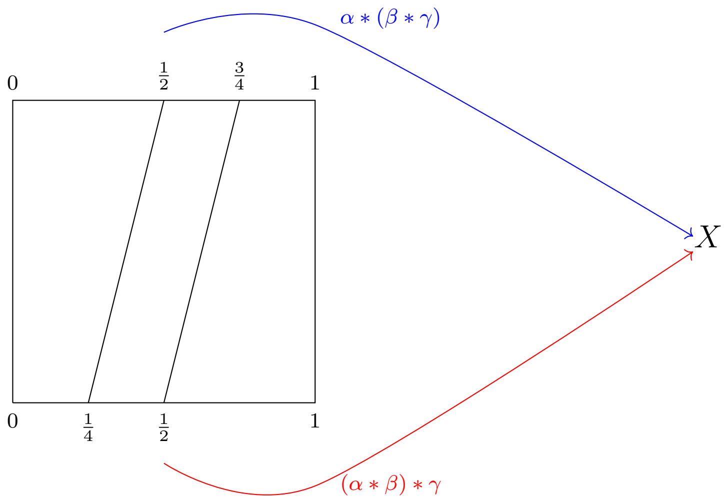
$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \dfrac{1}{2} \\ \beta(2s-1) & \text{if } \dfrac{1}{2} \leq s \leq 1 \end{cases}$$

Remark Note that $\alpha * \beta$ is continuous, maps $[0, \frac{1}{2}]$ onto the image of α in X, and maps $[\frac{1}{2}, 1]$ onto the image of β.

Unfortunately,this multiplication does not give a group structure on the set of loops based at a particular point since it is not even associative. For example, if α, β, γ are loops based at $\alpha(0)$, then

$$\alpha * (\beta * \gamma)(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(4s-2) & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4} \\ \gamma(4s-3) & \text{if } \frac{3}{4} \leq s \leq 1 \end{cases} \qquad (\alpha * \beta) * \gamma(s) = \begin{cases} \alpha(4s) & \text{if } 0 \leq s \leq \frac{1}{4} \\ \beta(4s-1) & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

although $\{\alpha*(\beta*\gamma)(s) \mid s \in [0,1]\} = \{(\alpha*\beta)*\gamma(s) \mid s \in [0,1]\}$, but, $\alpha*(\beta*\gamma)(s) \neq (\alpha*\beta)*\gamma(s)$



in general, i.e. the multiplication $*$ is not associative. In order to resolve this problem and obtain a group we shall identify two loops if one can be continuously deformed into the other, keeping the base point fixed throughout the deformation.

Definition If $f, g : X \to Y$ are continuous maps of the space X into space Y, we say that f is homotopic to g if there is a continuous map $F : X \times I \to Y$, where $I = [0,1]$, such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x) \quad \text{for each } x \in X.$$

The map F is called a homotopy between f and g. If f is homotopic to g, we write $f \underset{F}{\simeq} g$ or simply $f \simeq g$.

We think of a homotopy as a continuous one-parameter family of maps from X to Y. If we imagine the parameter t as representing time, then the homotopy F represents a continuous "deforming" of the map f to the map g, as t goes from 0 to 1.

Now we consider the special case in which f is a path in X. Recall that if $f : [0, 1] \to X$ is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$, we say that f is a path in X from the initial point x_0 to the final point x_1. In the following, we shall for convenience use the interval $I = [0, 1]$ as the domain for all paths.

If f and g are two paths in X, there is a stronger relation between them than mere homotopy. It is defined as follows:

Definition Two paths $f, g : I \to X$, mapping the interval $I = [0, 1]]$ into the space X are said to be path homotopic if they have the same initial point x_0 and the same final point x_1 and there is a continuous map $F : I \times I \to X$ such that

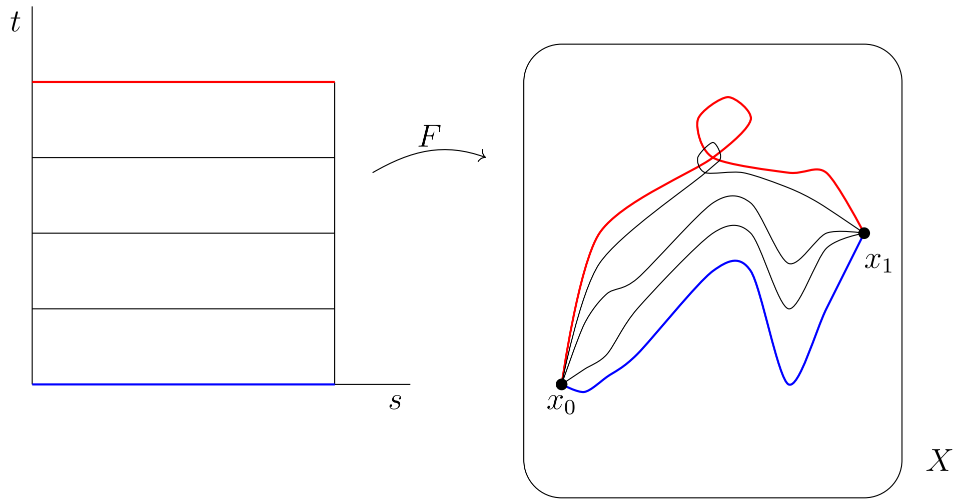
$$F(s, 0) = f(s) \quad \text{and} \quad F(s, 1) = g(s),$$
$$F(0, t) = x_0 \quad \text{and} \quad F(1, t) = x_1,$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and g. If f is path homotopic to g, we write $f \simeq_p g$.

The first condition says simply that F is a homotopy between f and g, and the second says that for each t, the path

$$s \to F(s, t)$$

is a path from x_0 to x_1.



In general, we say that f is homotopic to g relative to A (a subset of X) and write $f \simeq g$ rel A if there is a homotopy F from f to g with the additional property that

$$F(a, t) = f(a) \quad \text{for all } a \in A, \text{ for all } t \in I.$$

So, f is path homotopic to g if and only if f is homotopic to g relative to $\{0, 1\}$.

Examples

1. Let X be a topological space, C be a convex subset of the euclidean space $\mathbb{R}^n$ and let $f, g : X \to C$ be continuous maps. Then the map $F : X \times I \to C$ defined by

$$F(x, t) = (1 - t)f(x) + tg(x) \quad \text{for each } x \in X,\ t \in I = [0, 1]$$

is a homotopy (called straight-line homotopy) from f to g.

2. Let X be a topological space, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = \sum_{i=1}^{n+1} x_i^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$, and let $f, g : X \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be continuous maps such that $f(x) + g(x) \neq 0 \in \mathbb{R}^n$ for all $x \in X$. Then the map $F : X \times I \to \mathbb{S}^n$ defined by

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a homotopy from f to g.

Pasting Lemma (or Glueing Lemma) Let $f : X \to Y$ be a function mapping space X to space Y. Let $A \cup B = X$, where A and B are both open or both closed subsets of X. If $f|_A$ and $f|_B$ are both continuous, then f is continuous.

Proof Let U be an open subset of Y. We have that $f|_A^{-1}(U)$ is open in A and $f|_B^{-1}(U)$ is open in B. There exist open sets V_A and V_B of X such that $f|_A^{-1}(U) = A \cap V_A$ and $f|_B^{-1}(U) = B \cap V_B$. Since

$$f^{-1}(U) = f|_A^{-1}(U) \cup f|_B^{-1}(U) = (A \cap V_A) \cup (B \cap V_B),$$

and if both A and B are open subsets of X, then $f^{-1}(U) = (A \cap V_A) \cup (B \cap V_B)$ is open in X and f is continuous. It is also easy to show that f is continuous if sets A and B are closed.

Lemma The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Proof Let us verify the properties of an equivalence relation.

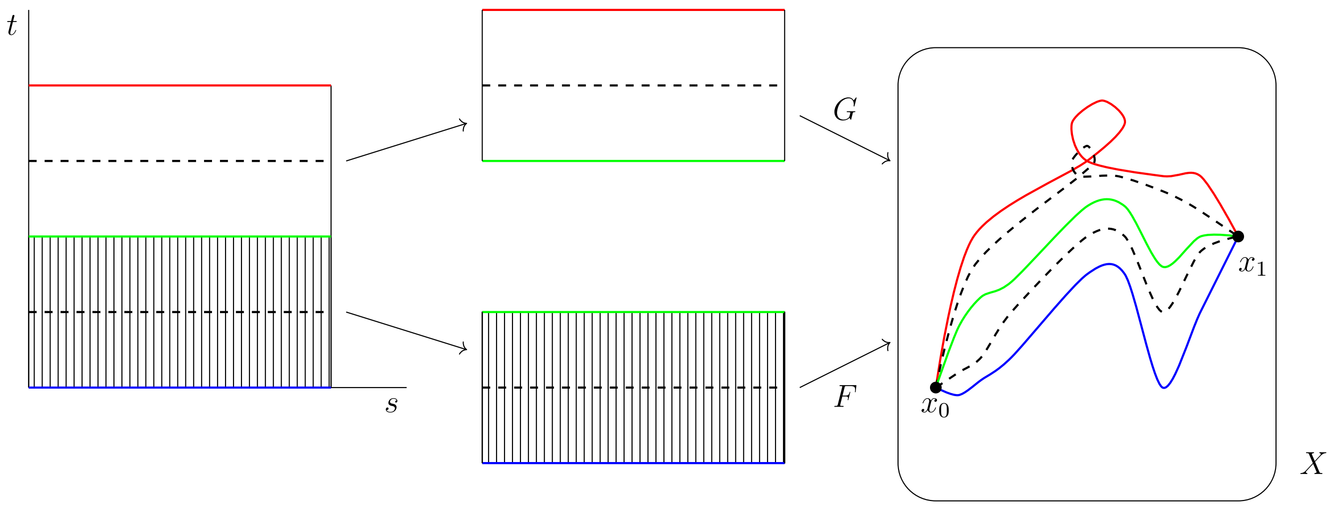
Given f, it is trivial that $f \simeq f$ since the map $F(x,t) = f(x)$ is the required homotopy. If f is a path, F is a path homotopy.

If $f \simeq g$ and if F is a homotopy between f and g, then $G(x,t) = F(x, 1-t)$ is a homotopy between g and f, that is $g \simeq f$. If F is a path homotopy, so is G.

If $f \simeq g$, $g \simeq h$ and if F is a homotopy between f and g, G is a homotopy between g and h, then the map $H : X \times I \to Y$, defined by

$$H(x,t) = \begin{cases} F(x, 2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ G(x, 2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is a homotopy between f and h, that is $f \simeq h$. If F and G are path homotopies, so is H.



Definition If f is a path in X from x_0 to x_1, and g is a path in X from x_1 to x_2, we define the composition $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The function h is well defined and continuous and it is a path in X from x_0 to x_2. We think of h as the path whose first half is the path f and whose second half is the path g.

We shall show that the operation of compositions on paths induces a well-defined operation $*$ on path-homotopy classes $[f]$ and $[g]$, so we can define

$$[f] * [g] = [f * g].$$

Furthermore, the operation $*$ on path-homotopy classes turns out to satisfy properties that look very much like the axioms for a group. The only difference from the properties of a group is that $[f] * [g]$ is not defined for every pair of classes, but only for those pairs $[f]$, $[g]$ for which $f(1) = g(0)$.

Theorem The operation $*$ is well-defined on path-homotopy classes. It has the following properties:

(1) (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$ and they are equal.

(2) (Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

(3) (Inverse) Given a path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. It is called the reverse of f. Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \text{and} \quad [\bar{f}] * [f] = [e_{x_1}].$$

Proof To show the operation well-defined, let F be a path homotopy between f and f', and let G be a homotopy between g and g'. Define

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,t) & \text{for } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Because $F(1,t) = x_1 = G(0,t)$ for all t, the map H is well-defined, continuous and it is a homotopy between $f * g$ and $f' * g'$.

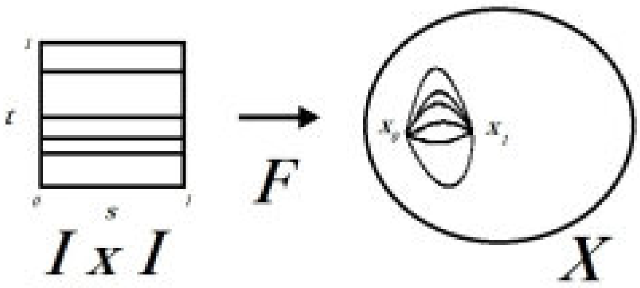
To prove the associativity, we need to show that $f * (g * h) \simeq_p (f * g) * h$. Since

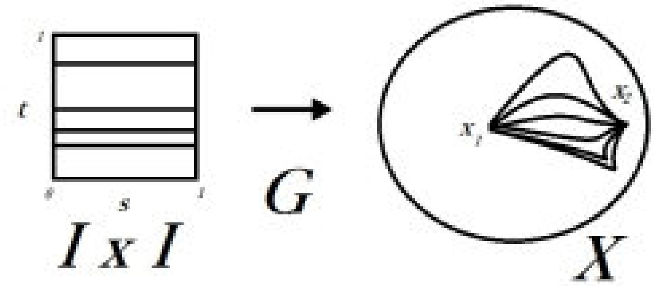
one can check that the map

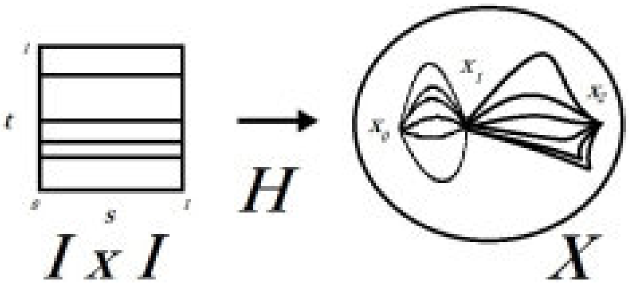
$$F(s,t) = \begin{cases} f(4s/(t+1)) & \text{for } s \in [0, (t+1)/4], \\ g(4s-t-1) & \text{for } s \in [(t+1)/4, (t+2)/4], \\ h((4s-t-2)/(2-t)) & \text{for } s \in [(t+2)/4, 1] \end{cases}$$

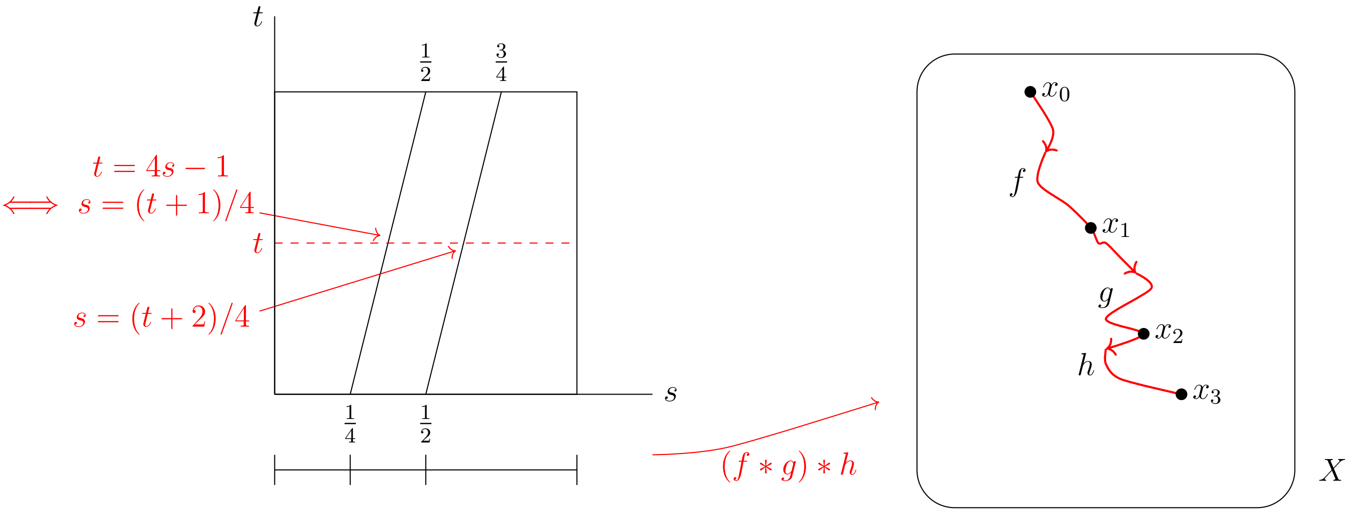
is a path homotopy between $(f * g) * h$ and $f * (g * h)$.

To show that $f \simeq_p f * e_{x_1}$, one can check that the map









$$G(s,t) = \begin{cases} f(2s/(2-t)) & \text{for } s \in [0,(2-t)/2], \\ x_1 & \text{for } s \in [(2-t)/2, 1] \end{cases}$$

is a path hompotpy between f and $f * e_{x_1}$.

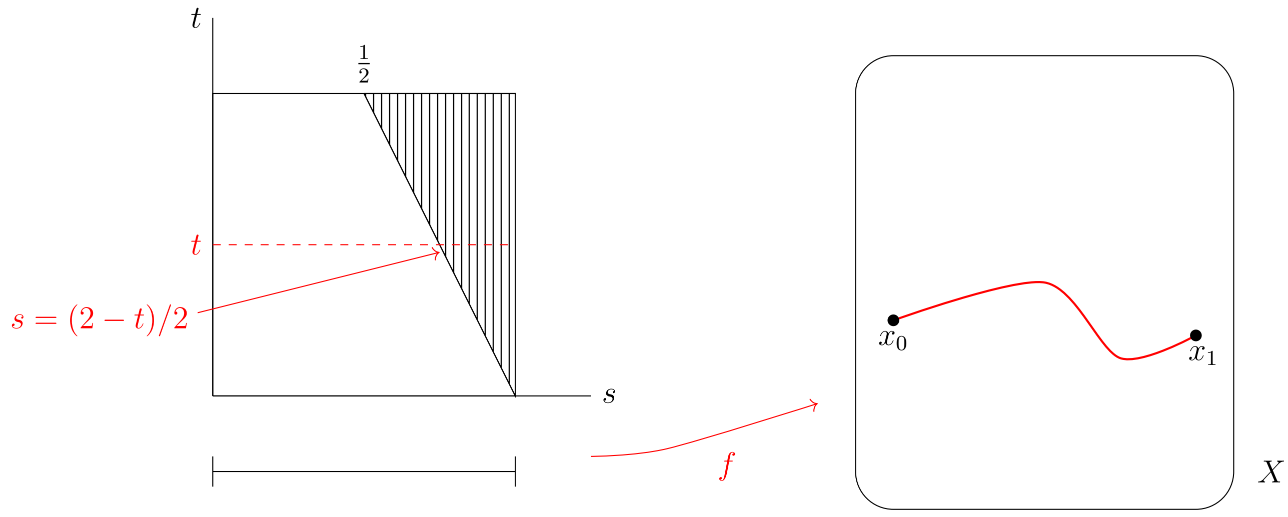
The proof of $e_{x_0} * f \simeq_p f$ is similar.

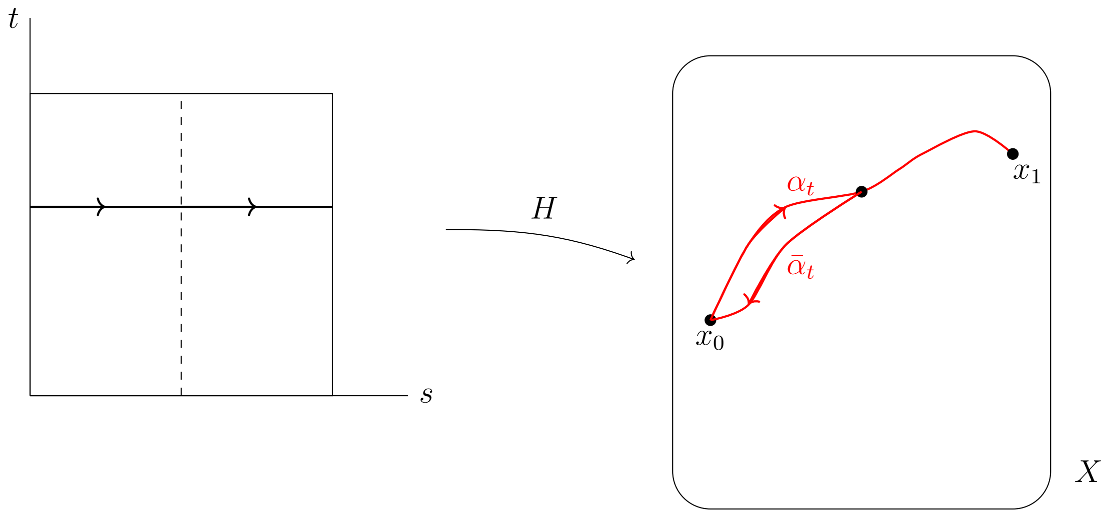
To show that $f * \bar{f} \simeq_p e_{x_0}$, one can check that the map

$$H(s,t) = \begin{cases} f(2ts) & \text{for } s \in [0,1/2], \\ f(2t(1-s)) & \text{for } s \in [1/2, 1] \end{cases}$$

is a path hompotpy between e_{x_0} and $f * \bar{f}$.

A similar argument could be used to show that $\bar{f} * f \simeq_p e_{x_1}$. But better yet, note the following: We have shown that for any path g, we have $g * \bar{g} \simeq_p e_x$, where x is the initial point of g.





In particular, $\bar{f} * \bar{\bar{f}} \simeq_p e_{x_1}$, where $\bar{\bar{f}}$ is the reverse of $\bar{f}$. But the reverse of $\bar{f}$ is just f! Thus $\bar{f} * f \simeq_p e_{x_1}$, as desired.

The Fundamental Group

Definition Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.

Examples

1. Let $\mathbb{R}^n$ denote the euclidean n-space. Then $\pi_1(\mathbb{R}^n, x_0) = \{e_{x_0}\}$, i.e $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group (the consisting of identity alone). For if f is a loop in $\mathbb{R}^n$ based at x_0, the straight-line homotopy
$$F(s,t) = tx_0 + (1-t)f(s)$$
is a path homotopy between f and the constant loop e_{x_0}.
2. More generally, if X is any convex subset of $\mathbb{R}n$, then $\pi_1(X, x_0)$ is the trivial group. The straight-line homotopy will work once again, for convexity of X means that for any x, $y \in X$,

the straight-line segment

$$\{tx + (1-t)y \mid 0 \le t \le 1\}$$

between them lies in X. In particular, the unit ball B^n in $\mathbb{R}^n$,

$$B^n = \{x \mid x_1^2 + \cdots + x_n^2 \le 1\},$$

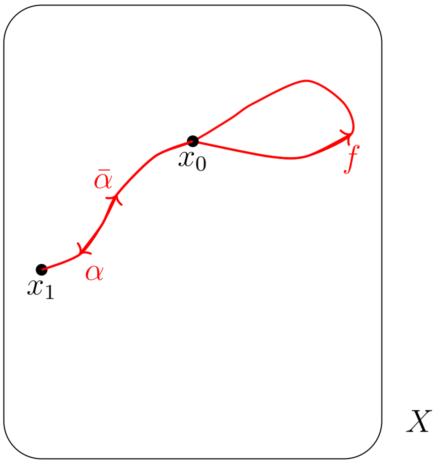
has trivial fundamental group.

Definition Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$



The map $\hat{\alpha}$ is well-defined because the operation $*$ is well defined. If f is a loop based at x_0, then $\bar{\alpha} * (f * \alpha)$ is a loop based at x_1. Hence $\hat{\alpha}$ maps $\pi_1(X, x_0)$ into $\pi_1(X, x_1)$, as desired.

Theorem The map $\hat{\alpha}$ is a group isomorphism.

Proof For any $[f]$, $[g] \in \pi_1(X, x_0)$, since

$$\begin{aligned}
\hat{\alpha}([f]) * \hat{\alpha}([g]) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\
&= ([\bar{\alpha}] * [f] * [g] * [\alpha]) \quad \text{since } \alpha * \bar{\alpha} \simeq_p e_{x_0} \text{ and } e_{x_0} * g \simeq_p g \\
&= \hat{\alpha}([f] * [g])
\end{aligned}$$

the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a homomorphism.

Let β denote the path $\bar{\alpha}$, i.e. the reverse of α, and let $\hat{\beta}$ be defined by

$$\hat{\beta}([h]) = [\bar{\beta}] * [h] * [\beta] = [\alpha] * [h] * [\bar{\alpha}] \quad \text{for each } [h] \in \pi_1(X, x_1).$$

Since

$$\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] = [h] \quad \text{for all } [h] \in \pi_1(X, x_1),$$

and $\hat{\beta}(\hat{\alpha}([f]))$ for all $[f] \in \pi_1(X, x_0)$, the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism.

Corollary If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Remark If X is path connected, all the groups $\pi_1(X, x)$ are isomorphic, so it is tempting to try to "identify" all these groups with one another, and to speak simply of the fundamental group of the space X, without reference to base point. The difficulty with this approach is that there is no natural way of identifying $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$; different paths α and β from x_0 to x_1 may give rise to different isomorphisms between these groups. For this reason, omitting the base point can lead to error.

Theorem The isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ is independent of path if and only if the fundamental group $\pi_1(X, x_0)$ is abelian.

Proof Let α and β be two paths from x_0 to x_1, and let $\hat{\alpha}$ and $\hat{\beta}$ be the group isomorphisms defined by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha] \quad \text{and} \quad \hat{\beta}([f]) = [\bar{\beta}] * [f] * [\beta] \quad \text{for all } [f] \in \pi_1(X, x_0)$$

Since

$$\hat{\beta}^{-1}(\hat{\alpha}([f])) = [\beta] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\beta}] = [\beta * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] \quad \text{for all } [f] \in \pi_1(X, x_0),$$

so for each $[f] \in \pi_1(X, x_0)$,

$$\hat{\alpha}([f]) = \hat{\beta}([f]) \iff [\beta * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] = \hat{\beta}^{-1}(\hat{\alpha}([f])) = [f] \iff [\beta * \bar{\alpha}] * [f] = [f] * [\beta * \bar{\alpha}]$$

if and only if $\pi_1(X, x_0)$ is abelian.

Definition A space X is said to be simply connected if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$.

We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof Let f and g be two paths from x_0 to x_1. Then $f * \bar{g}$ is defined and is a loop on X based at x_0. Since X is simply connected, $f * \bar{g} \simeq_p e_{x_0}$. Applying the groupoid properties, we see that

$$[(f * \bar{g}) * g] = [e_{x_0} * g] = [g].$$

But

$$[(f * \bar{g}) * g] = [f * (\bar{g} * g)] = [f * e_{x_1}] = [f].$$

Thus f and g are path homotopic.

Definition Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map that carries the point x_0 of X to the point y_0 of Y. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f] \quad \text{for each loop } f \text{ in } X \text{ based at } x_0.$$

The map h_* is called the homomorphism induced by h, relative to the base point x_0.

Remark It is easy to see that h_* is well defined. If f and f' are path homotopic, let $F : I \times I \to X$ be the homotopy between them. Then $h \circ F$ is a path homotopy between the loops $h \circ f$ and $h \circ f'$.

It is easy to check that h_* is a homomorphism. For

$$(f * g)(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s - 1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

It follows that

$$h((f*g)(s)) = \begin{cases} h(f(2s)) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ h(g(2s-1)) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Thus $h(f*g)$ equals the composition $(h \circ f) * (h \circ g)$. It follows that

$$h_*([f]*[g]) = h_*([f]) * h_*([g]),$$

so that h_* is a homomorphism.

Theorem If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous maps carry respectively $x_0 \in X$ to $y_0 \in Y$ and $y_0 \in Y$ to $z_0 \in Z$, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof The proof is a triviality. By definition,

$$\begin{aligned} (k \circ h)_*([f]) &= [(k \circ h) \circ f] \\ (k_* \circ h_*)([f]) &= k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)] \end{aligned}$$

Similarly, $i_*([f]) = [i \circ f] = [f]$.

Corollary If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y carries $x_0 \in X$ to $y_0 \in Y$, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Proof Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h. Then

$$k_* \circ h_* = (k \circ h)_* = i_*,$$

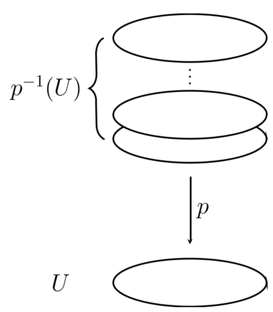
where i is the identity map of (X, x_0); and

$$h_* \circ k_* = (h \circ k)_* = j_*,$$

where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphisms of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, k_* is the inverse of h_*.

Covering Spaces

Definition Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be evenly covered by p if the inverse image $p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$ is the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices.



Definition Let $p : E \to B$ be continuous and surjective. If every point b of B has an open neighborhood U_b that is evenly covered by p, then p is called a covering map, and E is said to be a covering space of B.

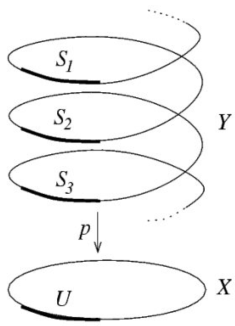
Remark Note that if $p : E \to B$ is a covering map, then for each $b \in B$ the subset $p^{-1}(b)$ of E necessarily has the discrete topology. For each slice V_α is open in E and intersects $p^{-1}(b)$ in a single point; therefore this point is open in the subspace topology on $p^{-1}(b)$.

Exmaples

1. Let X be any space; let $i : X \to X$ be the identity map. Then i is a covering map (of the most trivial sort). More generally let E be the space $X \times \{1, \ldots, n\}$ consisting of n disjoint copies of X. The map $p : E \to X$ given by $p(x, i) = x$ for all $1 \le i \le n$ is again a (rather trivial) covering map.
2. The map $p : \mathbb{R} \to \mathbb{S}^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.



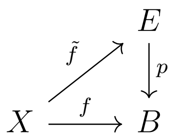
3. Consider the space $T = \mathbb{S}^1 \times \mathbb{S}^1$; it is called the torus. It is a general fact that the product of two covering maps

$$p \times p : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$$

is a covering map. Note that each of the unit squares $[n, n+1] \times [m, m+1]$ gets wrapped by $p \times p$ entirely around the torus.

The Fundamental Group of the Circle

Definition Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a lifting of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.



Example Consider the covering $p : \mathbb{R} \to \mathbb{S}^1$ defined by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x).$$

The path $f : [0, 1] \to \mathbb{S}^1$ beginning at $b_0 = (1, 0)$ given by $f(s) = (\cos \pi s, \sin \pi s)$ lifts to the path $\tilde{f}(s) = \dfrac{s}{2}$ beginning at 0 and ending at $\dfrac{1}{2}$.

The path $g(s) = (\cos \pi s, -\sin \pi s)$ lifts to the path $\tilde{g}(s) = -\dfrac{s}{2}$ beginning at 0 and ending at $-\dfrac{1}{2}$.

The path $h(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\tilde{h}(s) = 2s$ beginning at 0 and ending at 2.

Lemma Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

Proof We define the lifting $\tilde{f}$ step by step.

Step 1 : For each $b \in B$, let U_b be an open neighborhood of b that is evenly covered by p. Since $f([0,1])$ is a compact subset of B and $f([0,1]) \subseteq B = \bigcup_{b \in B} U_b$, there exists a subdivision of $[0,1]$, say $0 = s_0 < s_1 < \cdots < s_n = 1$, such that for each i the set $f([s_i, s_{i+1}]) \subset U_i$ for some $U_i \in \{U_b \mid b \in B\}$.

Step 2 : For each $0 \le i < n$, suppose that

- $\tilde{f}(s)$ is defined for $0 \le s \le s_i$,
- $f([s_i, s_{i+1}])$ lies in some open set U_i of B that is evenly covered by p,
- $p^{-1}(U_i)$ is the union of disjoint open sets $\{V_{i,\alpha}\}$ in E such that for each α, the restriction of p to $V_{i,\alpha}$ is a homeomorphism from $V_{i,\alpha}$ onto U_i by p,
- $\tilde{f}(s_i)$ lies in one of $\{V_{i,\alpha}\}$, say in $V_{i,0}$.

Define $\tilde{f} : [s_i, s_{i+1}] \to E$ by

$$\tilde{f}(s) = \left(p|_{V_{i,0}}\right)^{-1}(f(s)) \quad \text{for each } s \in [s_i, s_{i+1}].$$

Since $p|_{V_{i,0}} : V_{i,0} \to U_i$ is a homeomorphism and $f : [s_i, s_{i+1}] \to B$ is continuous, $\tilde{f}$ will be continuous on $[s_i, s_{i+1}]$.

Continuing in this way, we define $\tilde{f}$ on all of $[0,1]$. Continuity of $\tilde{f}$ follows from the pasting lemma; the fact that $p \circ \tilde{f} = f$ is immediate from the definition of $\tilde{f}$.

Further since

- the set $\tilde{f}([s_i, s_{i+1}])$ is connected,
- the slices $\{V_{i,\alpha}\}$ are open and disjoint, $\tilde{f}(s_i) \in V_{i,0}$
- $p \circ \tilde{f}(s) = f(s)$ for all $s \in [s_i, s_{i+1}]$, for each $0 \le i < n$,

$\tilde{f}([s_i, s_{i+1}])$ is lies entirely in $V_{i,0}$, and $\tilde{f}$ is the unique lifting of f (beginning at e_0).

Lemma Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a lifting of F to a continuous map

$$\tilde{F} : I \times I \to E$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof Given F, we first define $\tilde{F}(0,0) = e_0$. Next, we use the preceding lemma to extend $\tilde{F}$ to the left-hand edge $0 \times I$ and the bottom edge $I \times 0$ of $I \times I$. Then we extend $\tilde{F}$ to all of $I \times I$ as follows:

Step 1 : Use the Lebesgue number lemma to choose subdivisions

$$0 = s_0 < s_1 < \cdots < s_m = 1, \quad 0 = t_0 < t_1 < \cdots < t_n = 1$$

of I fine enough that each rectangle

$$I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$$

is mapped by F into an open set U_{ij} of B that is evenly covered by p.

Step 2 : We define the lifting $\tilde{F}$ step by step, beginning with the rectangle $I_1 \times J_1$, continuing with the other rectangles $I_i \times J_1$ in the "bottom row", then with the rectangles $I_i \times J_2$ in the next row, and so on.

Given $0 < i_0 < m$, $0 < j_0 < n$, suppose that

- $\tilde{F}$ is defined on
$$A = (0 \times I) \cup (I \times 0) \cup \left(\bigcup_{j<j_0} \bigcup_{i=1}^{m} I_i \times J_j \right) \cup \left(\bigcup_{i<i_0} I_i \times J_{j_0} \right)$$
- $F(I_{i_0} \times J_{j_0})$ lies in some open set $U_{i_0 j_0}$ of B that is evenly covered by p,
- $p^{-1}(U_{i_0 j_0})$ is the union of disjoint open sets $\{V_{i_0 j_0, \alpha}\}$ in E such that for each α, the restriction of p to $V_{i_0 j_0, \alpha}$ is a homeomorphism from $V_{i_0 j_0, \alpha}$ onto $U_{i_0 j_0}$ by p,
- $\tilde{F}(A \cap (I_{i_0} \times J_{j_0}))$ lies in one of $\{V_{i_0 j_0, \alpha}\}$, say in $V_{i_0 j_0, 0}$.

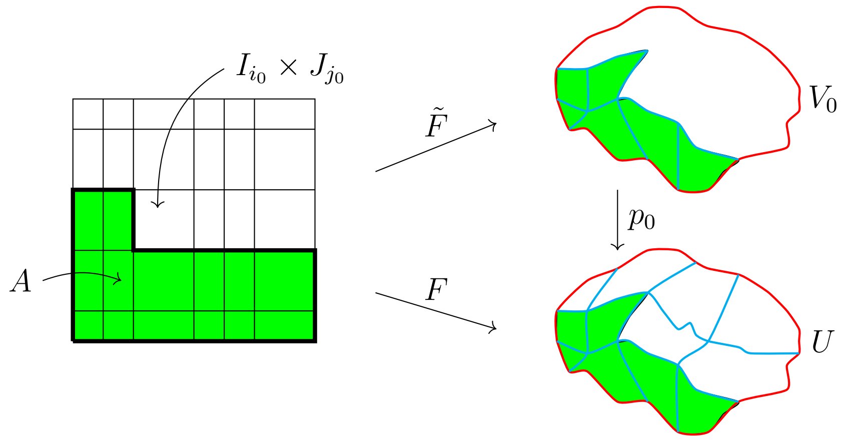
Define $\tilde{F} : I_{i_0} \times J_{j_0} \to E$ by

$$\tilde{F}(s,t) = \left(p|_{V_{i_0 j_0, 0}}\right)^{-1} (F(s,t)) \quad \text{for each } (s,t) \in I_{i_0} \times J_{j_0} = [s_{i_0-1}, s_{i_0}] \times [t_{j_0-1}, t_{j_0}].$$

Since $p|_{V_{i_0 j_0, 0}} : V_{i_0 j_0, 0} \to U_{i_0 j_0}$ is a homeomorphism and $F : I_{i_0} \times J_{j_0} \to B$ is continuous, $\tilde{F}$ will be continuous on $I_{i_0} \times J_{j_0}$.

Continuing in this way, we define $\tilde{F}$ on all of $I \times I$. Continuity of $\tilde{F}$ follows from the pasting lemma.

Suppose that $F : I \times I \to B$ is a path homotopy between paths f and g from b_0 to b_1 in B. Then $F(0 \times I) = \{b_0\}$ and $F(1 \times I) = \{b_1\}$, so its lifting $\tilde{F}$ of F must satisfy that $\tilde{F}(0 \times I) \subseteq p^{-1}(b_0)$ and $\tilde{F}(1 \times I) \subseteq p^{-1}(b_1)$. Furthermore, since $\tilde{F}$ is continuous, $0 \times I$ is connected and $p^{-1}(b_0)$ has discrete topology as a subspace of E, $\tilde{F}(0 \times I)$ is connected and thus must equal to a one-point set. Similarly. $\tilde{F}(1 \times I)$ must be a one-point set. Thus $\tilde{F}$ is a path homotopy.



Theorem Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1; let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Proof Let $F : I \times I \to B$ be a path homotopy between f and g. Then $F(0,0) = b_0$. Let $\tilde{F} : I \times I \to E$ be a lifting of F to E such that $\tilde{F}(0,0) = e_0$. By the preceding lemma, $\tilde{F}$ is a path homotopy, so that $\tilde{F}(0 \times I) = \{e_0\}$ and $\tilde{F}(1 \times I)$ is a one-point set $\{e_1\}$.

The restriction $\tilde{F}|_{I\times 0}$ of $\tilde{F}$ to the bottom edge of $I \times I$ is a path on E beginning at e_0 that is a lifting of $F|_{I\times 0}$. By uniqueness of path liftings, we must have $\tilde{F}(s,0) = \tilde{f}(s)$.

Similarly, $\tilde{F}|_{I\times 1}$ is a path on E that is a lifting of $F|_{I\times 1}$, and it begins at e_0 because $\tilde{F}(0 \times I) = \{e_0\}$. By uniqueness of path liftings, $\tilde{F}(s,1) = \tilde{g}(s)$.

Therefore, both $\tilde{f}$ and $\tilde{g}$ end at e_1, and $\tilde{F}$ is a path homotopy between them.

Definition Let $p : E \to B$ be a covering map; let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

We call ϕ the lifting correspondence derived from the covering map p. It depends of course on the choice of the point e_0.

Theorem The fundamental group $(\pi_1(S^1), *)$ of the circle is isomorphic to $(\mathbb{Z}, +)$.

Proof Let

- $p : \mathbb{R} \to \mathbb{S}^1$ be a covering map defined by

$$p(x) = (\cos 2\pi x, \sin 2\pi x) \quad \text{for } x \in \mathbb{R},$$

- $b_0 = (1, 0) \in \mathbb{S}^1$, f be a loop on $\mathbb{S}^1$ based at b_0,
- $\tilde{f}$ be a lifting of f to a path on $\mathbb{R}$ beginning at 0.

Since the point $\tilde{f}(1) \in p^{-1}(b_0) = \mathbb{Z}$, $\tilde{f}(1) = n$ for some integer $n \in \mathbb{Z}$. So, $\tilde{f}$ is the unique lifting of f on $\mathbb{R}$ from 0 to n, and we can define $\phi : \pi_1(\mathbb{S}^1, b_0) \to \mathbb{Z}$ by

$$\phi([f]) = \tilde{f}(1) = n.$$

Claim : ϕ is a group isomorphism between groups $(\pi_1(\mathbb{S}^1, b_0), *)$ and $(\mathbb{Z}, +)$.

The map ϕ is surjective. For each $n \in p^{-1}(b_0) = \mathbb{Z}$, since $\mathbb{R}$ is path connected, we can choose a path $\tilde{f} : [0,1] \to \mathbb{R}$ in $\mathbb{R}$ from 0 to n. Define $f = p \circ \tilde{f}$. Then f is a loop in $\mathbb{S}^1$ based at b_0, and $\tilde{f}$ is its lifting to a path in $\mathbb{R}$ beginning at 0. By definition, $\phi([f]) = n$.

The map ϕ is injective. Assume that $\phi([f]) = n = \phi([g])$, and let $\tilde{f}$ and $\tilde{g}$ be liftings of f and g, respectively, to paths on $\mathbb{R}$ beginning at 0; both $\tilde{f}$ and $\tilde{g}$ end at n, by hypothesis. Because $\mathbb{R}$ is simply connected, $\tilde{f}$ and $\tilde{g}$ are path homotopic; let $\tilde{F}$ be the path homotopy between them. The map $F = p \circ \tilde{F}$ will be a path homotopy between f and g, and we have $[f] = [g]$.

The map ϕ is a homomorphism. Let f and g be loops on $\mathbb{S}^1$ based at b_0; let $\tilde{f}$ and $\tilde{g}$ be their liftings, respectively, to paths on $\mathbb{R}$ beginning at 0. Let $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$. Define a path h on $\mathbb{R}$ by

$$h(s) = \begin{cases} \tilde{f}(2s) & \text{for } s \in [0, \frac{1}{2}], \\ n + \tilde{g}(2s-1) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

Then h is a path on $\mathbb{R}$ beginning at 0 and it is a lifting of $f * g$ on $\mathbb{R}$ beginning at 0 since

$$p(h(s)) = \begin{cases} p(\tilde{f}(2s)) = f(2s) & \text{for } s \in [0, \frac{1}{2}], \\ p(n + \tilde{g}(2s-1)) = p(\tilde{g}(2s-1)) = g(2s-1) & \text{for } s \in [\frac{1}{2}, 1], \end{cases} = (f * g)(s)$$

By definition

$$\phi([f] * [g]) = \phi([f * g]) = h(1) = n + m = \phi([f]) + \phi([g]) \implies \phi([f] * [g]) = \phi([f]) + \phi([g]).$$

Definition Let G be a group; let x be an element of G. we denote the inverse of x by x^{-1}. The symbol x^n denotes the n-fold product of x with itself, x^{-n} denotes the n-fold product of x^{-1} with itself, and x^0 denotes the identity element of G. If the set of all elements of the form x^m, for $m \in \mathbb{Z}$, equals G, then G is said to be a cyclic group, and x is said to be a generator of G.

The cardinality of a group is also called the order of the group. A group is cyclic of infinite order if and only if it is isomorphic to the additive group of integers; it is cyclic of order k if and only if it is isomorphic to the group $\mathbb{Z}/k$ of integers modulo k. The preceding theorem implies that the fundamental group of the circle is infinite cyclic.

Note that if x is a generator of the infinite cyclic group G, and if y is an element of the arbitrary group H, then there is a unique homomorphism h of G into H such that $h(x) = y$; it is defined by setting $h(x^n) = y^n$ for all n.

Theorem Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$.

(a) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism, i.e. an injective homomorphism.

(b) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets $\pi_1(B, b_0)/H = \{H * [g] \mid [g] \in \pi_1(B, b_0)\}$ of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

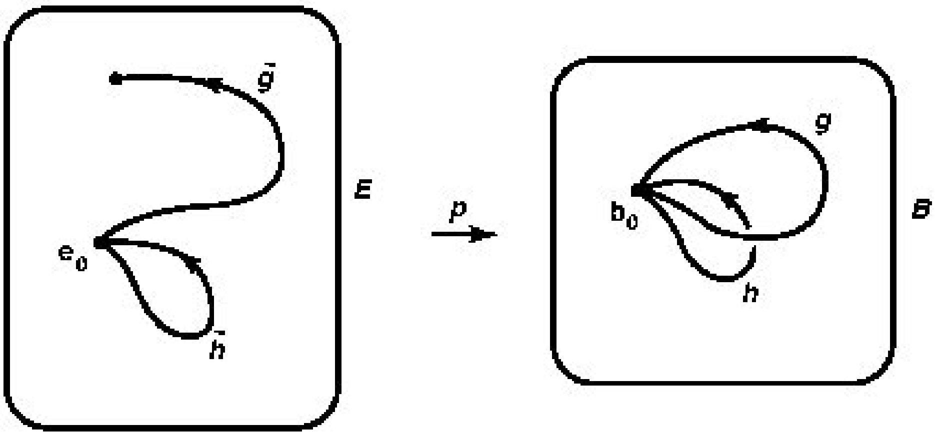
$$\begin{array}{c} p^{-1}(b_0) \\ \phi \big\uparrow \\ \pi_1(B, b_0) \end{array} \qquad \begin{array}{c} \phi([f]) = \tilde{f}(1) \in p^{-1}(b_0) \\ \phi \big\uparrow \\ [f] \in \pi_1(B, b_0) \end{array} \qquad \begin{array}{c} \Phi([f]) = \phi([f]) = \tilde{f}(1) = \tilde{h} * \tilde{g}(1) = \tilde{g}(1) = \phi([g]) \\ \Phi \big\uparrow \\ [f] = [h * g] \in H * [g] \subset \pi_1(B, b_0)/H \end{array}$$

(c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

Proof (*a*) Suppose $\tilde{h}$ is a loop in E at e_0, and $p_*([\tilde{h}])$ is the identity element ($[e_{b_0}]$). Let F be a path homotopy between $p \circ \tilde{h}$ and the constant loop e_{b_0}. If $\tilde{F}$ is the lifting of F to E such that $\tilde{F}(0, 0) = e_0$, then $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop e_{e_0} at e_0.

(*b*) Given loops f and g in B, let $\tilde{f}$ and $\tilde{g}$ be liftings of them to E that begin at e_0. Then $\phi([f]) = \tilde{f}(1)$ and $\phi([g]) = \tilde{g}(1)$. To show the map $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ is injective is equivalent to show that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$.

First, suppose that $[f] \in H * [g]$. Then $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Now the product $\tilde{h} * \tilde{g}$ is defined, and it is a lifting of $h * g$. Because $[f] = [h * g]$, the liftings $\tilde{f}$ and $\tilde{h} * \tilde{g}$, which begin at e_0, must end at the same point of E. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E, so that $\phi([f]) = \phi([g])$.



Now suppose that $\phi([f]) = \phi([g])$. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E. The product of $\tilde{f}$ and the reverse of $\tilde{g}$ is defined, and it is a loop $\tilde{h}$ in E based at e_0. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy in E between the loops $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy in B between $h * h$ and f, where $h = p \circ \tilde{h}$. Thus $[f] \in H * [g]$, as desired.

If E is path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective, so that Φ is surjective as well.

(c) Injectivity of Φ means that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$. Applying this result in the case where g is the constant loop, we see that $\phi([f]) = e_0$ if and only if $[f] \in H$. But $\phi([f]) = e_0$ precisely when the lift of f that begins at e_0 also ends at e_0.

Corollary Let $p : (E, e_0) \to (B, b_0)$ be a covering map. If E is path connected, then there is a surjection

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

If E is simply connected, ϕ is a bijection.

Theorem If G acts as a group of homeomorphisms on a simply connected space X, and if each point $x \in X$ has an open neighborhood U which satisfies $U \cap g(U) = \emptyset$ for all $g \in G \setminus \{e\}$, then $\pi_1(X/G)$ is isomorphic to G.

proof Fix a point $x_0 \in X$ and, given $g \in G$, join x_0 to $g(x_0)$ by a path γ. If $p : X \to X/G$ denotes the projection, $p \circ \gamma$ is a loop based at $p(x_0) \in X/G$.

Given $y \in X/G$, choose a point $x \in p^{-1}(y)$ and an open neighborhood U of x in X such that $U \cap g(U) = \emptyset$ for all $g \in G \setminus \{e\}$. If we set $V = p(U)$ and take $V_g = g(U)$ for each $g \in G$, then

- each $V_g = g(U)$ is homeomorphic to V, so it is open in X;
- V is open in X/G since $p : X \to X/G$ is a quotient map and $p^{-1}(V) = \bigcup_{g \in G} V_g$ is open in X.

This shows that $p : X \to X/G$ is a covering map and $\pi_1(X/G)$ is isomorphic to G since X is simply connected.

$$\begin{array}{ccccc} X & \quad & O(x) = \{g(x) \mid g \in G\} \subset X & \quad & \bigcup_{\{V_g = g(U) \mid g \in G\}} V_g \subset X \\ \Big\downarrow p & & \Big\downarrow p & & \Big\downarrow p \\ X/G & & \{x\} \subset X & & V = p(U) \subseteq X/G \end{array}$$

Remark In the proof, we have shown that the quotient map $p : X \to X/G$ is an open map, so it is a covering map. In particular, we have proved the following lemma.

Lemma If G acts as a group of homeomorphisms on a topological space X, the projection map $p : X \to X/G$ is an open map, where $X/G = \{O(x) \mid x \in X\}$ is a quotient space of X induced by p, and the subset $O(x) = \{g(x) \mid g \in G\}$ is called the orbit of x.

Proof For each $g \in G$ and any subset $U \subseteq X$, we define a set $g(U) \subseteq X$ by

$$g(U) = \{g(x) \mid x \in U\} \implies p^{-1}(p(U)) = \bigcup_{g\in G} g(U).$$

If $U \subseteq X$ is open, since each g is a homeomorphism from X onto X, each $g(U)$ is open, and therefore $p^{-1}(p(U)) = \bigcup_{g\in G} g(U)$ is open in X. Because p is a quotient map, this implies that $p(U)$ is open in X/G, and therefore p is an open map.

Definition A surface is a Hausdorff space with a countable basis, every point of which has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^2$.

Definition The projective plane $\mathbb{P}^2$ is the space obtained from $\mathbb{S}^2$ by identifying each point x of $\mathbb{S}^2$ with its antipodal point $-x$.

Theorem The projective plane $\mathbb{P}^2$ is a surface, and the map $p : \mathbb{S}^2 \to \mathbb{P}^2$ is a covering map.

Corollary $\pi_1(\mathbb{P}^2, y)$ is a group of order 2.

Definition If E is a simply connected space, and if $p : E \to B$ is a covering map, then we say that E is a universal covering space of B.

The Fundamental Group of the Punctured Plane

Theorem Let $x_0 \in \mathbb{S}^1$. The inclusion mapping

$$j : (\mathbb{S}^1, x_0) \to (\mathbb{R}^2 \setminus \mathbf{0}, x_0), \quad \text{where } \mathbf{0} = (0,0) \in \mathbb{R}^2$$

induces an isomorphism of fundamental groups.

Proof Let $r : \mathbb{R}^2 \setminus \mathbf{0} \to \mathbb{S}^1$ be the continuous map defined by

$$r(x) = \frac{x}{\|x\|}, \quad \text{where } \|x\| = \text{ the Euclidean distance from } x \text{ to the origin } \mathbf{0}.$$

The map r can be pictured as collapsing each radial ray in $\mathbb{R}^2 \setminus \mathbf{0}$ onto the point where the ray intersects $\mathbb{S}^1$; it maps each point x of $\mathbb{S}^1$ to itself.

Claim $r_* : \pi_1\left(\mathbb{R}^2 \setminus \mathbf{0}, x_0\right) \to \pi_1\left(\mathbb{S}^1, x_0\right)$ is an inverse for $j_* : \pi_1\left(\mathbb{S}^1, x_0\right) \to \pi_1\left(\mathbb{R}^2 \setminus \mathbf{0}, x_0\right)$.

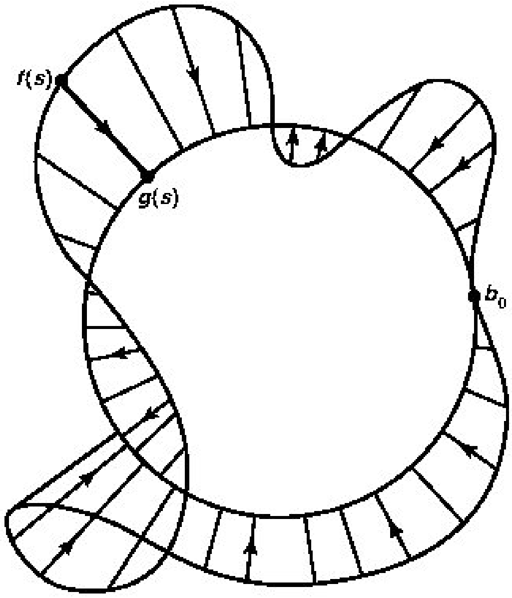
First consider the composite map

$$(\mathbb{S}^1, x_0) \xrightarrow{j} (\mathbb{R}^2 \setminus \mathbf{0}, x_0) \xrightarrow{r} (\mathbb{S}^1, x_0);$$

Since $r \circ j(x) = x$ for each $x \in \mathbb{S}^1$, $r_* \circ j_*$ is the identity isomorphism of $\pi_1(\mathbb{S}^1, x_0)$.

On the other hand, let f be a loop in $\mathbb{R}^2 \setminus \mathbf{0}$ based at x_0, and let g be a loop in $\mathbb{S}^1$ based at x_0 defined by

$$g(s) = j \circ r \circ f(s) = \frac{f(s)}{\|f(s)\|} \quad \text{for } s \in I.$$



Since the map $F : I \times I \to \mathbb{R}^2 \setminus \mathbf{0}$ defined by

$$F(s,t) = t\,\frac{f(s)}{\|f(s)\|} + (1-t)\,f(s)$$

satisfies that

$$\frac{t}{\|f(s)\|} + (1-t) \neq 0 \quad \text{and} \quad f(s) \neq 0 \implies F(s,t) \neq \mathbf{0} \quad \text{for all } s,\, t \in I,$$

and $F(0,t) = F(1,t) = x_0$ for all $t \in I$, F is a path homotopy between f and g and $j_* \circ r_*$ is the identity isomorphism of $\pi_1(\mathbb{R}^2 \setminus \mathbf{0}, x_0)$ with itself.

So, r_* is an inverse for j_* and $\pi_1(\mathbb{S}^1, x_0)$ is isomorphic to $\pi_1(\mathbb{R}^2 \setminus \mathbf{0}, x_0)$.

Theorem If $x_0 \in \mathbb{S}^{n-1}$, the inclusion mapping

$$j : (\mathbb{S}^{n-1}, x_0) \to (\mathbb{R}^n \setminus \mathbf{0}, x_0), \quad \text{where } \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$$

induces an isomorphism of fundamental groups.

Definition Let A be a subspace of X. Then A is said to be a strong deformation retract of X if there is a continuous map $H : X \times I \to X$ such that

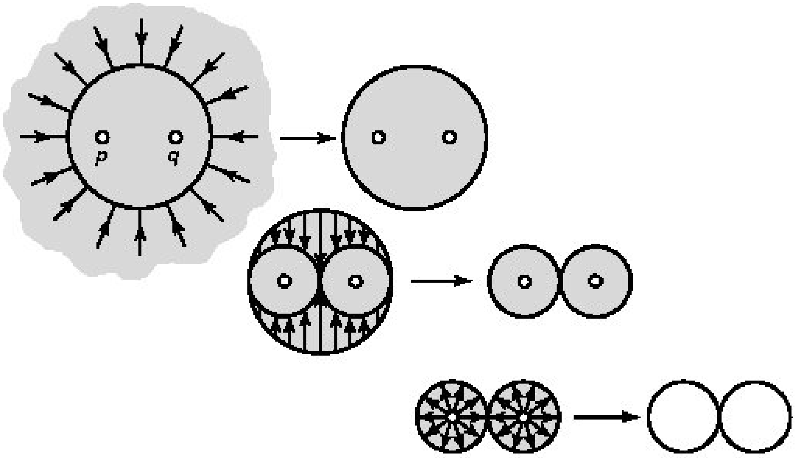
$$\begin{aligned} H(x,0) &= x && \text{for } x \in X \\ H(x,1) &\in A && \text{for } x \in A \\ H(a,t) &= a && \text{for } a \in A \text{ and } t \in I \end{aligned}$$

The map H is called a strong deformation retraction.

Example The map $H : \mathbb{R}^n \setminus \mathbf{0} \times I \to \mathbb{R}^n \setminus \mathbf{0}$ defined by

$$H(x,t) = t\,\frac{x}{\|x\|} + (1-t)\,x$$

is a strong deformation retract of $\mathbb{R}^n \setminus \mathbf{0}$ onto $\mathbb{S}^{n-1}$; it gradually collapses each radial line into the point where it intersects $\mathbb{S}^{n-1}$.



Example The following illustration shows that "figure eight" is a strong deformation retract of the doubly punctured plane $\mathbb{R} \setminus \{p, q\}$.

Theorem Let A be a strong deformation retract of X, and let $a_0 \in A$. Then the inclusion map

$$j : (A, a_0) \to (X, a_0)$$

induces an isomorphism of fundamental groups.

The proof is similar to that of the preceding Theorem of $\pi_1(\mathbb{S}^1, x_0) \simeq \pi_1(\mathbb{R}^2 \setminus \mathbf{0}, x_0)$.

The Fundamental Group of $\mathbb{S}^n$

Van Kampen Theorem Let $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is path connected, and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms

$$i : (U, x_0) \to (X, x_0) \quad \text{and} \quad j : (V, x_0) \to (X, x_0)$$

generate $\pi_1(X, x_0)$.

Proof This theorem states that, given any loop f in X based at x_0, it is path homotopic to a product of the form $(g_1 * (g_2 * (\cdots * g_n)))$, where each g_i is a loop in X based at x_0 that lies either in U or in V.

Let $f : I \to X$ be a loop based at x_0. We wish to show that f is path homotopic to a constant loop.

Step 1. By the Lebesgue number lemma, there is a subdivision

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

of the interval $[0, 1]$ such that for each i, the set $f([a_{i-1}, a_i])$ lies entirely in either U or V. Among all such subdivisions, choose one for which the number n of subintervals is minimal. Then it follows that for each i, the point $f(a_i) \in U \cap V$.

Suppose that $f(a_i) \notin U$, for instance. Then neither $f([a_{i-1}, a_i])$ nor $f([a_i, a_{i+1}])$ lies entirely in U. Therefore, both of them must lie entirely in V. We can then discard a_i from the subdivision, and still have a subdivision of $[0, 1]$ for which the image of each subinterval lies either in U or in V. This contradicts to minimality. Hence $f(a_i)$ must belong to U.

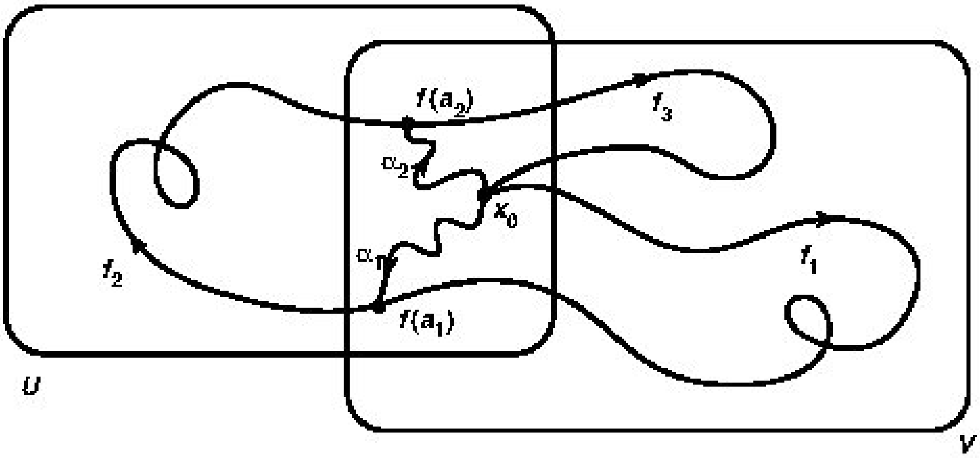
Step 2. Let f_i be the restriction of f to the interval $[a_{i-1}, a_i]$ defined by

$$f_i(s) = f((1-s)a_{i-1} + sa_i) \quad \text{for } s \in [0,1].$$

Then f_i is a path that lies either in U or in V, and

$$[f] = [f_1] * [f_2] * \cdots * [f_n]$$

For each i, choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$ (Here we use the fact that $U \cap V$ is path connected.) Since $f(a_0) = f(a_n) = x_0$, we can choose α_0 and α_n to be the constant path at x_0.



Now we set

$$g_i = (\alpha_{i-1} * f_i) * \bar{\alpha}_i \quad \text{for each } i.$$

Then g_i is a loop in X based at x_0 whose image lies either in U or in V. Direct computation shows that

$$[g_1] * [g_2] * \cdots * [g_n] = [f_1] * [f_2] * \cdots * [f_n].$$

Corollary (The special Van Kampen theorem) Suppose $X = U \cup V$, where U and V are open sets of X; suppose $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.

Theorem For $n \geq 2$, the n-sphere $\mathbb{S}^n$ is simply connected.

Proof Let $p = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the "north pole" of $\mathbb{S}^n$; let $q = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the "south pole" of $\mathbb{S}^n$.

Step 1. For each $x = (x_1, \ldots, x_{n+1}) \neq p$, $x \in \mathbb{S}^n$, let the stereographic projection map $f : \mathbb{S}^n \setminus p \to \mathbb{R}^n$ be define by

$$f(x) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$

Obviously f is continuous and it is in fact a homeomorphism since the map $g : \mathbb{R}^n \to \mathbb{S}^n \setminus p$ given by

$$g(y_1, \ldots, y_n) = (ty_1, ty_2, \ldots, ty_n, 1-t), \quad \text{where } t = \frac{2}{1 + y_1^2 + \cdots + y_n^2}$$

is an inverse for f. So, $\mathbb{S}^n \setminus p$ is homeomorphic to $\mathbb{R}^n$.

Since $\mathbb{S}^n \setminus q$ is homeomorphic to $\mathbb{S}^n \setminus p$ under the reflection map

$$p(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1}),$$

$\mathbb{S}^n \setminus q$ is also homeomorphic to $\mathbb{R}^n$.

Hence both $\mathbb{S}^n \setminus p$ and $\mathbb{S}^n \setminus p$ are simply connected since $\mathbb{R}^n$ is simply connected.

Step 2. Let $U = \mathbb{S}^n \setminus p$ and $V = \mathbb{S}^n \setminus q$. Since U and V are simply connected open sets of $\mathbb{S}^n$ such that $U \cup V = \mathbb{S}^n$, and since $U \cap V \simeq \mathbb{R}^n \setminus \mathbf{0}$ is path connected, $\mathbb{S}^n$ is simply connected by the special Van Kampen theorem.

Corollary $\mathbb{R}^n \setminus \mathbf{0}$ is simply connected if $n > 2$.

Proof If $n > 2$ and if $x_0 \in \mathbb{S}^{n-1}$, since $\mathbb{S}^{n-1}$ is simply connected and, by a preceding theorem, the inclusion map

$$j : (\mathbb{S}^{n-1}, x_0) \to (\mathbb{R}^n \setminus \mathbf{0}, x_0)$$

induces an isomorphism of fundamental groups, $\mathbb{R}^n \setminus \mathbf{0}$ is simply connected.

Corollary $\mathbb{R}^n$ and $\mathbb{R}^2$ are not homeomorphic for $n > 2$.

Proof Deleting a point from $\mathbb{R}^n$ leaves a simply connected space, while deleting a point from $\mathbb{R}^2$ does not.

Fundamental Groups of Surfaces

Theorem Let X, Y be topological spaces, and let $x_0 \in X$, $y_0 \in Y$. Then $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Remark Recall that if A and B are groups with operation $\cdot$, then the Cartesian product $A \times B$ is given a group structure by using the operation

$$(a \times b) \cdot (a' \times b') = (a \cdot a') \times (b \cdot b').$$

Recall also that if $h : C \to A$ and $k : C \to B$ are group homomorphisms, then the map $\Phi : C \to A \times B$ defined by $\Phi(c) = h(c) \times k(c)$ is a group homomorphism.

Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection mappings, $p_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0)$ and $q_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0)$ be the induced homomorphisms. Then we define a homomorphism

$$\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by the equation

$$\Phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \circ f].$$

Let $g : I \to X$ be a loop in X based at x_0, and let $h : I \to Y$ be a loop in Y based at y_0. Since the map $f : I \to X \times Y$ defined by

$$f(s) = g(s) \times h(s), \quad \text{for } s \in I$$

is a loop in $X \times Y$ based at $x_0 \times y_0$ and satisfies that

$$\Phi([f]) = [p \circ f] \times [q \circ f] = [g] \times [h],$$

the map Φ is surjective.

Suppose that $f : I \to X \times Y$ is a loop in $X \times Y$ based at $x_0 \times y_0$ such that

$$\Phi([f]) = [e_{x_0}] \times [e_{y_0}].$$

Since $\Phi([f]) = [p \circ f] \times [q \circ f]$, we have $p \circ f \simeq_p e_{x_0}$ and $q \circ f \simeq_p e_{y_0}$; let G and H be the respective path homotopies. Then the map $F : I \times I \to X \times Y$ defined by

$$F(s,t) = G(s,t) \times H(s,t)$$

is a path homotopy between f and the constant loop based at $x_0 \times y_0$.

Hence Φ is an isomorphism.

Corollary The fundamental group of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Van Kampen's Theorem (revisit) Let X be a topological space and let $U, V \subset X$ be open subsets such that $U \cap V$ is nonempty and path-connected. Let $x \in U \cap V$ be a basepoint. Then

$$\pi_1(X,x) = \pi_1(U,x) *_{\pi_1(U \cap V)} \pi_1(V,x).$$

Here, $A *_C B$ denotes the amalgamated product. Suppose you have groups A, B, C and homomorphisms $f : C \to A$ and $g : C \to B$. In our case, $A = \pi_1(U,x)$, $B = \pi_1(V,x)$ and $C = \pi_1(U \cap V, x)$, the map f is the pushforward map i_* where $i : U \cap V \to U$ is the inclusion, and g is the pushforward j_* where $j : U \cap V \to V$ is the inclusion.

Given A, B, C, f, g, you can define the amalgamated product

$$A *_C B = \langle \text{generators of } A,\ \text{generators of } A \mid \text{relations of } A, \text{relations of } B, \text{amalgamated relations} \rangle.$$

The amalgamated relations come from elements $c \in C$: each $c \in C$ gives a relation $f(c) = g(c)$.

Amalgamated relations

What does this mean? We have $f(c) \in A$ and we have already in the presentation of $A *_C B$ the generators and relations of A, so I can make sense of the element $f(c)$ in the amalgamated product. Similarly, $g(c) \in B$ and we can think of this as an element in the amalgamated product. The corresponding amalgamated relation is just saying that these two elements agree.

Geometrically, in the context of Van Kampen's theorem, C consists of loops living in the intersection $U \cap V$; the amalgamated relations are then saying "you can think of these loops as living in U; you can think of them living in V; it makes no difference".

Examples

1. The 2-sphere X can be written as $U \cup V$ where U is a neighbourhood of the Northern hemisphere and V is a neighbourhood of the Southern hemisphere. The overlap $U \cap V$ is an annular neighbourhood of the equator. Van Kampen's theorem tells us that $\pi_1(X,x) = \pi_1(U,x) *_{\pi_1(U \cap V, x)} \pi_1(V,x)$.
   We have $\pi_1(U) = \pi_1(V) = \{1\}$ as both U and V are simply-connected discs. Since $U \cap V$ is homotopy equivalent to the circle, $\pi_1(U \cap V) = \mathbb{Z} = \langle c \rangle$ (i.e. one generator, c, and no relations).
   The amalgamated product $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ has the empty set of generators coming from $\pi_1(U)$ and the empty set of generators coming from $\pi_1(V)$. Therefore it has no generators and the amalgamated product is trivial.
   The fact that $\pi_1(U \cap V)$ is nontrivial doesn't affect this computation: it would only enter into the relations in the group. Indeed, for each $c \in \pi_1(U \cap V)$, we have $i_* c \in \pi_1(U)$ and $j_* c \in \pi_1(V)$, so the amalgamated relations all become $1 = 1$, which we don't need to include as a relation because it holds in any group.
   The upshot of all of this is the statement that $\pi_1(\mathbb{S}^2) = \{1\}$.

2. Let $X = S^1 \vee S^1$ be the wedge of two circles (i.e. the disjoint union of two circles modulo an equivalence relation which identifies one point on the first circle with one point on the second. We take U to be a neighbourhood of the first circle and V to be a neighbourhood of the second. The intersection $U \cap V$ is a cross-shaped neighbourhood of the point where the two circles intersect (the wedge point).

   Since $U \simeq S^1$ and $V \simeq S^1$ we have $\pi_1(U) = \pi_1(V) = \mathbb{Z}$. Moreover, $U \cap V$ is contractible, so $\pi_1(U \cap V) = \{1\}$. The amalgamated product has a generator a coming from $\pi_1(U)$ and a generator b coming from $\pi_1(V)$. There are no relations coming from $\pi_1(U)$,, none coming from $\pi_1(V)$ and also no amalgamated relations (since $\pi_1(U \cap V) = \{1\}$).

   A presentation for $\pi_1(S^1 \vee S^1)$ is therefore

   $$\langle a, b \rangle$$

   This has no relations: we call a group with no relations a free group on its generators. The elements of a free group $\langle a, b \rangle$ are words on its generators, written using as, bs, a^{-1}s, b^{-1}s, for example

   $$a^2 b^{-1} a^{14} bab^2.$$

   The only simplifications one may perform with these elements are things like $aa^{-1} = 1$ or $bb^{-1} = 1$. In particular, the free group on two generators is not abelian (the relation $ab = ba$ does not hold).

3. Let $X = \mathbb{T}^2$, thought of as a square with its opposite sides identified. Let U be an open disc in the middle of the square. Let V be (a small open thickening of) the complement of U. The intersection $U \cap V$ is a circle. We have

   - $\pi_1(U)$ is trivial, as U is a disc;
   - $\pi_1(U \cap V) = \mathbb{Z}$;
   - $\pi_1(V) = \langle a, b \rangle$, as $V \simeq S^1 \vee S^1$.

   To see $V \simeq S^1 \vee S^1$, note that the square minus a disc is homotopy equivalent to the boundary of the square, which becomes a wedge of two circles in the quotient space.

   Van Kampen's theorem then tells us

   $$\pi_1(\mathbb{T}^2) = \langle a, b \mid i_*(c) = j_*(c) \rangle,$$

   where

   - $c \in \mathbb{Z}$ is a generator,
   - $i_* : \mathbb{Z} \to \{1\}$ is the trivial map,
   - $j_* : \mathbb{Z} \to \langle a, b \rangle$ sends a generator for $\mathbb{Z}$ to some word in a, b

   This amalgamated relation $j_*(c) = 1$ is equivalent to the boundary of the circle is homotopic to the loop $b^{-1}a^{-1}ba = 1$ in V. Therefore the amalgamated relation for the generator $c \in \mathbb{Z}$ is $b^{-1}a^{-1}ba = 1$.

   This relation is equivalent to $ab = ba$, so the group we get is just the abelian group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.